

STRICHARTZ ESTIMATES ON KERR BLACK HOLE BACKGROUNDS

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ABSTRACT. We study the dispersive properties for the wave equation in the Kerr space-time with small angular momentum. The main result of this paper is to establish Strichartz estimates for solutions of the aforementioned equation. This follows a local energy decay result for the Kerr space-time obtained in the earlier work [23] and uses the techniques and results from [13] by the author and collaborators. As an application, we then prove global well-posedness and uniqueness for the energy critical semilinear wave equation.

1. INTRODUCTION

Understanding the decay properties of solutions to the linear wave equation on Kerr backgrounds is considered a crucial first step in proving the stability of the Kerr solution to the Einstein equations. Until recently even the problem of obtaining uniform bounds for such solutions was completely open, and only some partial results (pointwise decay and energy bounds for azimuthal solutions away from the event horizon) were obtained in [6], [7]. Recently in [23] and independently in [4], [5], [1] uniform pointwise bounds as well as local energy decay were established for small angular momentum. The aim of this paper is to prove Strichartz estimates under the same assumption of small angular momentum.

The Kerr geometry in Boyer-Lindquist coordinates is given by

$$ds^2 = g_{tt}dt^2 + g_{t\phi}dtd\phi + g_{rr}dr^2 + g_{\phi\phi}d\phi^2 + g_{\theta\theta}d\theta^2$$

where $t \in \mathbb{R}$, $r > 0$, (ϕ, θ) are the spherical coordinates on \mathbb{S}^2 and

$$\begin{aligned} g_{tt} &= -\frac{\Delta - a^2 \sin^2 \theta}{\rho^2}, & g_{t\phi} &= -2a \frac{Mr \sin^2 \theta}{\rho^2}, & g_{rr} &= \frac{\rho^2}{\Delta} \\ g_{\phi\phi} &= \frac{(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta}{\rho^2} \sin^2 \theta, & g_{\theta\theta} &= \rho^2 \end{aligned}$$

with

$$\Delta = r^2 - 2Mr + a^2, \quad \rho^2 = r^2 + a^2 \cos^2 \theta.$$

Here M represents the mass of the black hole, and aM its angular momentum.

A straightforward computation gives us the inverse of the metric:

$$g^{tt} = -\frac{(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta}{\rho^2 \Delta}, \quad g^{t\phi} = -a \frac{2Mr}{\rho^2 \Delta}, \quad g^{rr} = \frac{\Delta}{\rho^2},$$

$$g^{\phi\phi} = \frac{\Delta - a^2 \sin^2 \theta}{\rho^2 \Delta \sin^2 \theta}, \quad g^{\theta\theta} = \frac{1}{\rho^2}.$$

The case $a = 0$ corresponds to the Schwarzschild space-time. We shall subsequently assume that a is small $a \ll M$, so that the Kerr metric is a small perturbation of the Schwarzschild metric. We let $\square_{\mathbf{K}}$ denote the d'Alembertian associated to the Kerr metric.

In the above coordinates the Kerr metric has singularities at $r = 0$ on the equator $\theta = \pi/2$ and at the roots of Δ , namely $r_{\pm} = M \pm \sqrt{M^2 - a^2}$. The singularity at $r = r_+$ is just a coordinate singularity, and corresponds to the event horizon. The singularity at $r = r_-$ is also a coordinate singularity; for a further discussion of its nature, which is not relevant for our results, we refer the reader to [3, 9]. To remove the singularities at $r = r_{\pm}$ we introduce functions r^* , v_+ and ϕ_+ so that (see [9])

$$dr^* = (r^2 + a^2)\Delta^{-1}dr, \quad dv_+ = dt + dr^*, \quad d\phi_+ = d\phi + a\Delta^{-1}dr.$$

The metric then becomes

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)dv_+^2 + 2drdv_+ - 4a\rho^{-2}Mr\sin^2\theta dv_+d\phi_+ - 2a\sin^2\theta drd\phi_+ + \rho^2d\theta^2 \\ + \rho^{-2}[(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta] \sin^2\theta d\phi_+^2$$

which is smooth and nondegenerate across the event horizon up to but not including $r = 0$. Just like in [13] and [23], we introduce the function

$$\tilde{v} = v_+ - \mu(r)$$

where μ is a smooth function of r . In the $(\tilde{v}, r, \phi_+, \theta)$ coordinates the metric has the form

$$ds^2 = \left(1 - \frac{2Mr}{\rho^2}\right)d\tilde{v}^2 + 2\left(1 - \left(1 - \frac{2Mr}{\rho^2}\right)\mu'(r)\right)d\tilde{v}dr \\ - 4a\rho^{-2}Mr\sin^2\theta d\tilde{v}d\phi_+ + \left(2\mu'(r) - \left(1 - \frac{2Mr}{\rho^2}\right)(\mu'(r))^2\right)dr^2 \\ - 2a\theta(1 + 2\rho^{-2}Mr\mu'(r))\sin^2\theta drd\phi_+ + \rho^2d\theta^2 \\ + \rho^{-2}[(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta] \sin^2\theta d\phi_+^2.$$

On the function μ we impose the following two conditions:

- (i) $\mu(r) \geq r^*$ for $r > 2M$, with equality for $r > 5M/2$.
- (ii) The surfaces $\tilde{v} = \text{const}$ are space-like, i.e.

$$\mu'(r) > 0, \quad 2 - \left(1 - \frac{2Mr}{\rho^2}\right)\mu'(r) > 0.$$

As long as a is small, we can work with the same function μ as in the case of the Schwarzschild space-time.

For convenience we also introduce

$$\tilde{\phi} = \zeta(r)\phi_+ + (1 - \zeta(r))\phi$$

where ζ is a cutoff function supported near the event horizon and work in the $(\tilde{v}, r, \tilde{\phi}, \theta)$ coordinates which are identical to (t, r, ϕ, θ) outside of a small neighborhood of the event horizon.

Given $r_- < r_e < 2M$ we consider the wave equation

$$(1.1) \quad \square_{\mathbf{K}} u = f$$

in the cylindrical region

$$(1.2) \quad \mathcal{M}_R = \{\tilde{v} \geq 0, r \geq r_e\}$$

with initial data on the space-like surface

$$(1.3) \quad \Sigma_R^- = \mathcal{M}_R \cap \{\tilde{v} = 0\}$$

The lateral boundary of \mathcal{M}_R ,

$$(1.4) \quad \Sigma_R^+ = \mathcal{M}_R \cap \{r = r_e\}$$

is also space-like, and can be thought of as the exit surface for all waves which cross the event horizon.

We define the initial (incoming) energy on Σ_R^- as

$$(1.5) \quad E[u](\Sigma_R^-) = \int_{\Sigma_R^-} (|\partial_r u|^2 + |\partial_{\tilde{v}} u|^2 + |\nabla u|^2) r^2 dr d\omega$$

the outgoing energy on Σ_R^+ as

$$(1.6) \quad E[u](\Sigma_R^+) = \int_{\Sigma_R^+} (|\partial_r u|^2 + |\partial_{\tilde{v}} u|^2 + |\nabla u|^2) r_e^2 d\tilde{v} d\omega$$

and the energy on an arbitrary \tilde{v} slice as

$$(1.7) \quad E[u](\tilde{v}_0) = \int_{\mathcal{M}_R \cap \{\tilde{v} = \tilde{v}_0\}} (|\partial_r u|^2 + |\partial_{\tilde{v}} u|^2 + |\nabla u|^2) r^2 dr d\omega$$

For solutions to the constant coefficient wave equation on $\mathbb{R} \times \mathbb{R}^3$, the well-known Strichartz estimates state that

$$(1.8) \quad \| |D_x|^{-\rho_1} \nabla u \|_{L_t^{p_1} L_x^{q_1}} \lesssim \| \nabla u(0) \|_{L^2} + \| |D_x|^{\rho_2} f \|_{L_t^{p'_2} L_x^{q'_2}}.$$

Here the exponents (ρ_i, p_i, q_i) are subject to the scaling relation

$$(1.9) \quad \frac{1}{p} + \frac{3}{q} = \frac{3}{2} - \rho$$

and the dispersion relation

$$(1.10) \quad \frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}, \quad 2 < p \leq \infty.$$

All pairs (ρ, p, q) satisfying (1.9) and (1.10) are called Strichartz pairs. Those for which the equality holds in (1.10) are called sharp Strichartz pairs. Such estimates first appeared in the seminal works [2], [18, 19] and as stated include contributions from, e.g., [8], [16], [10], [12], and [11].

If one allows variable coefficients, such estimates are well-understood locally-in-time. For smooth coefficients, this was first shown in [15] and later for C^2 coefficients in [17] and [20, 21, 22].

Globally-in-time, the problem is more delicate. Even a small, smooth, compactly supported perturbation of the flat metric may refocus a group of rays and produce caustics. Thus, constructing a parametrix for incoming rays proves to be quite difficult. At the

same time, one needs to contend with the possibility of trapped rays at high frequencies and with eigenfunctions/resonances at low frequencies.

Our main theorem is the following global in time estimate:

Theorem 1.1. *If u solves $\square_{\mathbf{K}}u = f$ in \mathcal{M}_R then for all nonsharp Strichartz pairs (ρ_1, p_1, q_1) and (ρ_2, p_2, q_2) we have*

$$(1.11) \quad E[u](\Sigma_R^+) + \sup_{\tilde{v}} E[u](\tilde{v}) + \|\nabla u\|_{L_{\tilde{v}}^{p_1} \dot{H}_x^{-\rho_1, q_1}}^2 \lesssim E[u](\Sigma_R^-) + \|f\|_{L_{\tilde{v}}^{p'_2} \dot{H}_x^{\rho_2, q'_2}}^2.$$

Here the Sobolev-type spaces $\dot{H}^{s,p}$ coincide with the usual $\dot{H}^{s,p}$ homogeneous spaces in \mathbb{R}^3 expressed in polar coordinates (r, ω) .

As a corollary of this result one can consider the global solvability question for the energy critical semilinear wave equation in the Kerr space. Let

$$\Sigma_0 = \{t = 0\}$$

Note that Σ_0 is a smooth, spacelike surface.

We now consider the Cauchy problem with initial data on Σ_0 :

$$(1.12) \quad \begin{cases} \square_{\mathbf{K}}u = \pm u^5 & \text{in } \mathcal{M} \\ u = u_0, \tilde{T}u = u_1 & \text{in } \Sigma_0. \end{cases}$$

Here \tilde{T} is a vector field that is smooth, everywhere timelike and equals $\partial_{\tilde{v}}$ on Σ_0 outside \mathcal{M}_C . Observe that we cannot use $\partial_{\tilde{v}}$ on all of Σ_0 since it becomes spacelike inside the ergosphere (i.e. when $g_{tt} > 0$).

Theorem 1.2. *Let $r_e > r_-$. Then there exists $\epsilon > 0$ so that for each initial data (u_0, u_1) which satisfies*

$$E[u](\Sigma_0) \leq \epsilon$$

the equation (1.12) admits a unique solution u in the region $\{r > r_e\}$ which satisfies the bound

$$E[u](\Sigma_{r_e}) + \|u\|_{\dot{H}^{s,p}(\{r > r_e\})} \lesssim E[u](\Sigma_0)$$

for all indices s, p satisfying

$$\frac{4}{p} = s + \frac{1}{2}, \quad 0 \leq s < \frac{1}{2}.$$

Furthermore, the solution has a Lipschitz dependence on the initial data in the above topology.

Theorem 1.2 follows from Theorem 1.1 exactly like in Section 5 of [13], hence we will now focus on proving the latter.

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2. THE STRICHARTZ ESTIMATES

In this section we will prove Theorem 1.1. The key to the proof is the approach developed in [14] for small perturbations of the Minkowski metric and [13] for the Schwarzschild metric which allows one to establish global-in-time Strichartz estimates provided that a strong form of the local energy estimates holds.

A weaker local energy result has been proved in [23]. We first improve this to a stronger result that only requires logarithmic losses in the energy norm. We then apply the techniques from [14] and [13] to obtain Strichartz estimates for all nonsharp exponents.

Let us first recall the setup and results from [23]. Let τ, ξ, Φ and Θ be the Fourier variables corresponding to t, r, ϕ and θ , and

$$p(r, \phi, \tau, \xi, \Phi, \Theta) = g^{tt}\tau^2 + 2g^{t\phi}\tau\Phi + g^{\phi\phi}\Phi^2 + g^{rr}\xi^2 + g^{\theta\theta}\Theta^2$$

be the principal symbol of $\square_{\mathbf{K}}$. On any null geodesic one has

$$(2.1) \quad p(t, r, \phi, \theta, \tau, \xi, \Phi, \Theta) = 0.$$

Moreover, all trapped null geodesics in the exterior $r > r_+$ must also satisfy (see [23] for more details):

$$(2.2) \quad \xi = 0$$

$$(2.3) \quad r = \text{constant} \text{ so that } (2r\Delta - (r - M)\rho^2)^2 \leq 4a^2r^2\Delta\sin^2\theta$$

$$(2.4) \quad R_a(r, \tau, \Phi) = 0$$

where

$$\begin{aligned} R_a(r, \tau, \Phi) = & (r^2 + a^2)(r^3 - 3Mr^2 + a^2r + a^2M)\tau^2 \\ & - 2aM(r^2 - a^2)\tau\Phi - a^2(r - M)\Phi^2 \end{aligned}$$

By (2.1) we can bound Φ in terms of τ ,

$$(2.5) \quad |\Phi| \leq 4M|\tau|$$

For Φ in this range and small a the polynomial $\tau^{-2}R_a(r, \tau, \Phi)$ can be viewed as a small perturbation of

$$\tau^{-2}R_0(r, \tau, \Phi) = r^4(r - 3M)$$

which has a simple root at $r = 3M$. Hence for small a the polynomial R_a has a simple root close to $3M$, which we denote by $r_a(\tau, \Phi)$.

In [23] the following result is proved:

Theorem 2.1. *Let u solve $\square_{\mathbf{K}}u = f$ in \mathcal{M}_R . Then*

$$(2.6) \quad \|u\|_{LEW_{\mathbf{K}}}^2 + \sup_{\tilde{v}} E[u](\tilde{v}) + E[u](\Sigma_R^+) \lesssim E[u](\Sigma_R^-) + \|f\|_{LEW_{\mathbf{K}}^*}^2.$$

The exact definition of the norms $LEW_{\mathbf{K}}$ is not important; what matters is that it is equivalent to $H_{t,x}^1$ on compact sets outside an $O(1)$ neighborhood of $r = 3M$ and is degenerate on the trapped set (described above). Near infinity (say for $r > 4M$) the norm $LEW_{\mathbf{K}}$ is defined on dyadic regions as follows:

$$(2.7) \quad \|u\|_{LEW_{\mathbf{K}}} = \sup_{j \in \mathbb{Z}} 2^{-\frac{j}{2}} \|\nabla_{t,x} u\|_{L^2(\mathbb{R} \times \{|x| \in [2^{j-1}, 2^j]\})} + 2^{-\frac{3j}{2}} \|u\|_{L^2(\mathbb{R} \times \{|x| \in [2^{j-1}, 2^j]\})}$$

The norm $LEW_{\mathbf{K}}^*$ is the dual of $LEW_{\mathbf{K}}$; in particular, near infinity it will be defined as

$$(2.8) \quad \|f\|_{LEW_{\mathbf{K}}^*} = \sum_{j \in \mathbb{Z}} 2^{\frac{j}{2}} \|f\|_{L^2(\mathbb{R} \times \{|x| \in [2^{j-1}, 2^j]\})}$$

We first improve Theorem 2.1 around $r = 3M$. We will work with the usual Boyer-Lindquist coordinates (t, r, ϕ, θ) since we are only interested in improving the estimate around $r = 3M$. In these coordinates we can write

$$(2.9) \quad \rho^2 \sqrt{\Delta} \square_{\mathbf{K}} \frac{1}{\sqrt{\Delta}} = L_K = \Delta \partial_r^2 + \left(-\frac{(r^2 + a^2)^2}{\Delta} \partial_t^2 - a \frac{2Mr}{\Delta} \partial_t \partial_\phi - \frac{a^2}{\Delta} \partial_\phi^2 + L_a\right) + V(r),$$

where

$$L_a = a^2 \sin^2 \theta \partial_t^2 + \frac{1}{\sin^2 \theta} \partial_\phi^2 + \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta)$$

and

$$V(r) = \sqrt{\Delta} \partial_r (\Delta \partial_r \frac{1}{\sqrt{\Delta}})$$

We can now take the Fourier transform in t (this is allowed since by Theorem 2.1 since u is a tempered distribution in t) and ϕ , and expand $\mathcal{F}_t L_a$ (which is an elliptic operator on \mathbb{S}^2) as the countable sum of its eigenvectors and corresponding eigenvalues λ_a^2 (note that λ_0 correspond to the usual spherical harmonics). This is possible since the operators ∂_ϕ and L_a commute. We are left with the ordinary differential equation

$$(2.10) \quad (\Delta \partial_r^2 + V_{\lambda_a, \tau, \Phi}(r)) w(r) = g$$

where

$$(2.11) \quad V_{\lambda_a, \tau, \Phi}(r) = \tau^2 W(r - r_a(\tau, \Phi)) - \lambda_a^2 + V.$$

and $W(0) = W'(0) = 0$, $W''(0) > 0$.

Let $\gamma_0 : \mathbb{R} \rightarrow \mathbb{R}^+$ be a smooth increasing function so that

$$\gamma_0(y) = \begin{cases} 1 & y < 1 \\ y & y \geq 2. \end{cases}$$

Let $\gamma_1 : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a smooth increasing function so that

$$\gamma_1(y) = \begin{cases} y^{\frac{1}{2}} & y < 1/2 \\ 1 & y \geq 1. \end{cases}$$

Let $\gamma : \mathbb{R}^2 \rightarrow \mathbb{R}^+$ be a smooth function with the following properties:

$$(2.12) \quad \gamma(y, z) = \begin{cases} 1 & z \leq C \\ \gamma_0(y) & y < \sqrt{z/2}, z \geq C \\ z^{\frac{1}{2}} \gamma_1(y^2/z) & y \geq \sqrt{z/2}, z \geq C \end{cases}$$

where C is a large constant. In the sequel z is either a discrete parameter or very large (see Remark 2.2), so the lack of smoothness at $z = C$ is of no consequence.

We define the symbol

$$b_{ps}(r, \tau, \xi, \Phi, \lambda_a) = \gamma(-\psi(\frac{\lambda_a}{\tau}) \ln((r - r_a(\tau, \Phi))^2 + \lambda_a^{-2} \xi^2), \ln \lambda_a),$$

and its inverse

$$b_{ps}^{-1}(r, \tau, \xi, \Phi, \lambda_a) = \frac{1}{\gamma(-\psi(\frac{\lambda_a}{\tau}) \ln((r - r_a(\tau, \Phi))^2 + \lambda_a^{-2} \xi^2), \ln \lambda_a)}.$$

Here $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth cutoff such that

$$\psi(y) = \begin{cases} 1 & y \in [-4, -2] \cup [2, 4] \\ 0 & y \in (-\infty, 8] \cup [-1, 1] \cup [8, \infty) \end{cases}$$

The role of ψ is to make sure that $b = 1$ when $\tau \ll \lambda_a$ and $\tau \gg \lambda_a$.

Observe that, as opposed to their Schwarzschild counterparts a_{ps} and a_{ps}^{-1} defined in [13], the symbols b_{ps} and b_{ps}^{-1} will depend on τ and Φ .

We note that if λ_a is small then b_{ps} and b_{ps}^{-1} both equal 1, while if λ_a is large then they satisfy the bounds

$$(2.13) \quad \begin{aligned} 1 &\leq b_{ps}(r, \tau, \xi, \Phi, \lambda_a) \leq b_{ps}(r, \tau, 0, \Phi, \lambda_a) \leq (\ln \lambda_a)^{\frac{1}{2}}, \\ (\ln \lambda_a)^{-\frac{1}{2}} &\leq b_{ps}^{-1}(r, \tau, 0, \Phi, \lambda_a) \leq b_{ps}^{-1}(r, \tau, \xi, \Phi, \lambda_a) \leq 1. \end{aligned}$$

We also observe that the region where $y^2 > z$ corresponds to $(r - r_a(\tau, \Phi))^2 + \lambda_a^{-2} \xi^2 < e^{-\sqrt{\ln \lambda_a}}$. Thus differentiating the two symbols we obtain the following bounds

$$(2.14) \quad \begin{aligned} |\partial_r^\alpha \partial_\xi^\beta \partial_{\lambda_a}^\nu \partial_\tau^\eta b_{ps}(r, \tau, \xi, \Phi, \lambda_a)| &\leq c_{\alpha, \beta, \nu} (1 + |\ln((r - r_a(\tau, \Phi))^2 + \lambda_a^{-2} \xi^2)|) \lambda_a^{-\beta - \nu - \eta} \\ &\quad ((r - r_a(\tau, \Phi))^2 + \lambda_a^{-2} \xi^2 + e^{-\sqrt{\ln \lambda_a}})^{-\frac{\alpha + \beta + \eta}{2}}, \end{aligned}$$

respectively

$$(2.16) \quad \begin{aligned} |\partial_r^\alpha \partial_\xi^\beta \partial_{\lambda_a}^\nu \partial_\tau^\eta b_{ps}^{-1}(r, \tau, \xi, \Phi, \lambda_a)| &\leq c_{\alpha, \beta, \nu} b_{ps}^{-2}(r, \tau, \xi, \Phi, \lambda_a) (1 + |\ln((r - r_a(\tau, \Phi))^2 + \lambda_a^{-2} \xi^2)|) \\ &\quad \lambda_a^{-\beta - \nu - \eta} ((r - r_a(\tau, \Phi))^2 + \lambda_a^{-2} \xi^2 + e^{-\sqrt{\ln \lambda_a}})^{-\frac{\alpha + \beta + \eta}{2}}, \end{aligned}$$

when $\alpha + \beta + \nu + \tau > 0$ and $\ln \lambda \geq C$. These show that we have a good operator calculus for the corresponding pseudodifferential operators. In particular in terms of the classical symbol classes we have

$$(2.18) \quad b_{ps}, b_{ps}^{-1} \in S_{1-\delta, 0}^\delta, \quad \delta > 0.$$

By (2.15) and (2.17) one easily sees that these operators are approximate inverses. More precisely for small λ_a , $\ln \lambda_a < C$, they are both the identity, while for large λ_a

$$(2.19) \quad \|b_{ps}^w(\lambda_a)(b_{ps}^{-1})^w(\lambda_a) - I\|_{L^2 \rightarrow L^2} \lesssim \lambda_a^{-1} e^{\sqrt{\ln \lambda_a}}, \quad \ln \lambda_a \geq C.$$

Choosing C large enough we insure that the bound above is always much smaller than 1.

Remark 2.2. *The role played by λ_a changes from the proof of Theorems 2.3 to the proof of the Strichartz estimates. Since all of our L^2 estimates admit orthogonal decompositions with respect to the eigenfunctions associated to λ_a , it suffices for Theorems 2.3 to fix λ_a and work with the operators $b_{ps}^w(\lambda_a)$. However, in the proof of the Strichartz estimates we need kernel bounds for operators of the form B_{ps} , which is why we think of λ_a as a fourth Fourier variable (besides τ , Φ and ξ) and track the symbol regularity with respect to λ_a as well. Of course, this is meaningless for λ_a in a compact set; only the asymptotic behavior as $\lambda_a \rightarrow \infty$ is relevant.*

We can now introduce the Weyl operators

$$B_{ps} = \sum_{\lambda_a} b_{ps}^w(\lambda_a) \Pi_{\lambda_a},$$

respectively

$$B_{ps}^{-1} = \sum_{\lambda_a} (b_{ps}^{-1})^w(\lambda_a) \Pi_{\lambda_a}.$$

where Π_{λ_a} are the spectral projectors on the eigenspaces of L_a determined by λ_a .

We use these two operators in order to define the improved local energy norms around $r = 3M$. Let I be a small neighborhood of $3M$ (which does not depend on a). We say that $\tilde{u} \in LEK_{ps}$ if $\tilde{u} : \mathbb{R} \times I \times \mathbb{S}^2 \rightarrow \mathbb{R}$ and

$$(2.20) \quad \|\tilde{u}\|_{LEK_{ps}} = \|B_{ps}^{-1} \tilde{u}\|_{H^1} \approx \|A_{ps}^{-1} \nabla \tilde{u}\|_{L^2} < \infty,$$

Similarly $\tilde{f} \in LEK_{ps}^*$ if $\tilde{f} : \mathbb{R} \times I \times \mathbb{S}^2 \rightarrow \mathbb{R}$ and

$$(2.21) \quad \|\tilde{f}\|_{LEK_{ps}^*} = \|B_{ps} \tilde{f}\|_{L^2} < \infty.$$

Let $\chi(r)$ be a cutoff function supported on I which equals 1 on a smaller neighborhood of size $O(1)$ near $3M$, and $u : \mathcal{M}_R \rightarrow \mathbb{R}$. We say that $u \in LE_{\mathbf{K}}$ if χu is the restriction of some $\tilde{u} \in LEK_{ps}$ on \mathcal{M}_R , and

$$(2.22) \quad \|u\|_{LE_{\mathbf{K}}} = \inf_{\tilde{u}|_{\mathcal{M}_R} = \chi u} \|\tilde{u}\|_{LEK_{ps}} + \|(1 - \chi)u\|_{LEW_{\mathbf{K}}}$$

Similarly if $f : \mathcal{M}_R \rightarrow \mathbb{R}$, then $f \in LE_{\mathbf{K}}^*$ if χf is the restriction of some $\tilde{f} \in LEK_{ps}^*$ on \mathcal{M}_R , and

$$(2.23) \quad \|f\|_{LE_{\mathbf{K}}^*} = \inf_{\tilde{f}|_{\mathcal{M}_R} = \chi f} \|\chi \tilde{f}\|_{LEK_{ps}^*} + \|(1 - \chi)f\|_{LEW_{\mathbf{K}}^*}$$

Our improved local energy estimate reads:

Theorem 2.3. *For all functions u which solve $\square_{\mathbf{K}} u = f$ in \mathcal{M}_R we have*

$$\sup_{\tilde{v} > 0} E[u](\tilde{v}) + E[u](\Sigma_R^+) + \|u\|_{LE_{\mathbf{K}}}^2 \lesssim E[u](\Sigma_R^-) + \|f\|_{LE_{\mathbf{K}}^*}^2.$$

Proof. We start with the following estimate near $r = 3M$, which is the equivalent of Proposition 3.3 from [13]:

Proposition 2.4. *a) Let u be a function supported in $\{5M/2 < r < 5M\}$ which solves $\square_{\mathbf{K}} u = f$. Then*

$$(2.24) \quad \|u\|_{LEK_{ps}}^2 \lesssim \|f\|_{LEK_{ps}^*}^2.$$

b) Let $f \in LEK_{ps}^$ be supported in $\{11M/4 < r < 4M\}$. Then there is a function u supported in $\{5M/2 < r < 5M\}$ so that*

$$(2.25) \quad \sup_t E[u] + \|u\|_{LEK_{ps}}^2 + \|\square_{\mathbf{K}} u - f\|_{LE_{\mathbf{K}}^{1*}}^2 \lesssim \|f\|_{LEK_{ps}^*}^2.$$

Proof. We start with part (a). By Plancherel's formula and the fact that

$$(2.26) \quad \Phi < \lambda_a$$

(2.24) will follow if we can prove that

$$(2.27) \quad \|\partial_r w(r)\|_{L^2} + (|\tau| + |\lambda_a|) \|(b_{ps}^{-1})^w(\tau, \Phi, \lambda_a) w\|_{L^2} \lesssim \|b_{ps}^w(\tau, \Phi, \lambda_a) g\|_{L^2}.$$

for any fixed τ, Φ and λ_a (here $b_{ps}^w(\tau, \Phi, \lambda_a)$ is the one dimensional pseudodifferential operator obtained from b_{ps} by fixing τ, Φ and λ_a .)

Depending on the relative sizes of λ_a, τ and Φ and taking into account (2.26) we consider several cases. In the easier cases it suffices to replace the bound (2.27) with a simpler bound

$$(2.28) \quad \|\partial_r w\|_{L^2} + (|\tau| + |\lambda_a|) \|w\|_{L^2} \lesssim \|g\|_{L^2}.$$

Case 1: $\lambda_a, \tau, \Phi \lesssim 1$. Then $V_{\lambda_a, \tau, \Phi}(r) \approx 1$. We solve (2.10) as a Cauchy problem with data on one side; namely, if

$$E[w](r) = \Delta(\partial_r w)^2 + w^2$$

then an easy computation shows that

$$\frac{d}{dr} E[w](r) \lesssim E[w](r) + g^2$$

By Gronwall's inequality and the fact that $E[w](r) = 0$ away from the photonsphere we obtain the pointwise bound

$$|w| + |\partial_r w| \lesssim \|g\|_{L^2}$$

which easily implies (2.28).

Case 2: $\Phi, \lambda_a \ll \tau$. Then $V_{\lambda_a, \tau, \Phi}(r) \approx \tau^2$, $\partial_r V_{\lambda_a, \tau, \Phi}(r) \approx \tau^2$ for r in a compact set; therefore (2.10) is hyperbolic in nature. Hence we can solve (2.10) as a Cauchy problem with data on one side; namely, if

$$E[w](r) = \Delta(\partial_r w)^2 + V_{\lambda_a, \tau, \Phi}(r) w^2$$

then an easy computation shows that

$$\frac{d}{dr} E[w](r) \lesssim E[w](r) + g^2$$

By Gronwall's inequality and the fact that $E[w](r) = 0$ away from the photonsphere we obtain the pointwise bound

$$\tau|w| + |\partial_r w| \lesssim \|g\|_{L^2}$$

which implies (2.28).

Case 3: $\lambda_a \gg \tau$. Then $V_{\lambda_a, \tau, \Phi}(r) \approx -\lambda_a^2$ for r in a compact set; therefore (2.10) is elliptic. Then we solve (2.10) as an elliptic problem with Dirichlet boundary conditions on a compact interval and obtain

$$\lambda_a^{\frac{3}{2}} \|w\|_{L^2} + \lambda_a^{\frac{1}{2}} \|\partial_r w\|_{L^2} \lesssim \|g\|_{L^2}$$

which again gives (2.28).

Case 4: $\lambda_a \approx \tau \gg 1$. In this case (2.28) no longer holds. However, we can now use Proposition 3.4 from [13] which states

Proposition 2.5. *Let W be a smooth function satisfying $W(0) = W'(0) = 0$, $W''(0) > 0$, and $|\epsilon| \lesssim 1$. Let w be a solution of the ordinary differential equation*

$$(\partial_{r^*}^2 + \lambda^2(W(r^*) + \epsilon))w(r^*) = g$$

supported near $r^ = 0$. Then we have*

$$(2.29) \quad \|\partial_{r^*} u\|_{L^2} + \lambda \|(a_{ps}^{-1})^w(\lambda)u\|_{L^2} \lesssim \|a_{ps}^w(\lambda)g\|_{L^2},$$

Here the symbols $a_{ps}(\lambda)$ and $(a_{ps}^{-1})(\lambda)$ are defined as follows

$$a_{ps}(\lambda)(r^*, \xi) = \gamma(-\ln(r^{*2} + \lambda^{-2}\xi^2), \ln \lambda),$$

$$a_{ps}^{-1}(\lambda)(r^*, \xi) = \frac{1}{\gamma(-\ln(r^{*2} + \lambda^{-2}\xi^2), \ln \lambda)}.$$

We can now apply Proposition 2.5 with $r^* = r - r_a(\tau, \Phi)$, W as in (2.11) and with $b_{ps}^w(\tau, \Phi, \lambda_a)$ and $(b_{ps}^{-1})^w(\tau, \Phi, \lambda_a)$ replacing $a_{ps}^w(\lambda)$ and $(a_{ps}^{-1})^w(\lambda)$ respectively. Note that there the extra Δ coefficient in front of ∂_r^2 in (2.10) plays no role, since $\Delta \approx \partial_r \Delta \approx 1$ near $r = 3M$.

Part (b) follows now from part (a) exactly like in Proposition 3.3 from [13] with λ_a replacing the spherical harmonics λ and $r^* = r - r_a(\tau, \Phi)$. \square

\square

We now repeat the arguments in Section 4 of [13] to turn the improved local energy estimate, Proposition 2.4 into Strichartz estimates, Theorem 1.1. The only (minor) differences appear in proving Case II of Proposition 4.10, which we will settle below. We thus need to prove

Proposition 2.6. *For u supported in $\{5M/2 < r < 5M\}$ we have*

$$\|\nabla u\|_{L_v^p H^{-\rho, q}}^2 \lesssim E[u](0) + \|u\|_{LEK_{ps}}^2 + \|\square_K u\|_{LEK_{ps}^*}^2.$$

Proof. Clearly the operator \square_K can be replaced by L_K . The potential V can be neglected due to the straightforward bound

$$\|Vu\|_{LE_{ps}^*} \lesssim \|u\|_{LE_{ps}}.$$

Indeed, for u_{λ_a} the eigenfunction corresponding to the eigenvalue λ_a^2 we have

$$\|Vu_{\lambda_a}\|_{LEK_{ps}^*} \lesssim |\ln(2 + \lambda_a)|^{\frac{1}{2}} \|u_{\lambda_a}\|_{L^2} \lesssim \lambda_a |\ln(2 + \lambda_a)|^{-\frac{1}{2}} \|u_{\lambda_a}\|_{L^2} \lesssim \|u_{\lambda_a}\|_{LE_{ps}}$$

We introduce the auxiliary function

$$\psi = B_{ps}^{-1}u.$$

By the definition of the LEK_{ps} norm we have

$$(2.30) \quad \|\psi\|_{H^1} \lesssim \|u\|_{LEK_{ps}}.$$

We also claim that

$$(2.31) \quad \|L_K \psi\|_{L^2} \lesssim \|u\|_{LEK_{ps}} + \|L_K u\|_{L^2}.$$

Since B_{ps}^{-1} is L^2 bounded, this is a consequence of the commutator bound

$$[B_{ps}^{-1}, L_K] : LEK_{ps} \rightarrow L^2,$$

or equivalently

$$(2.32) \quad [B_{ps}^{-1}, L_K] B_{ps} : H^1 \rightarrow L^2.$$

It suffices to consider the first term in the symbol calculus, as the remainder belongs to $OPS_{1-\delta, \delta}^\delta$, mapping H^δ to L^2 for all $\delta > 0$. The symbol of the first term is

$$q(\tau, \Phi, \xi, r, \lambda_a) = \{b_{ps}^{-1}(\lambda_a), \Delta \xi^2 - \frac{(r^2 + a^2)^2}{\Delta} \tau^2 - a \frac{2Mr}{\Delta} \tau \Phi - \frac{a^2}{\Delta} \Phi^2 + \lambda_a^2\} b_{ps}(\lambda_a)$$

and a-priori we have $q \in S_{1-\delta, \delta}^{1+\delta}$. For a better estimate we compute the Poisson bracket

$$\begin{aligned} q(\tau, \Phi, \xi, r, \lambda_a) &= b_{ps}^{-1}(\lambda_a) \gamma_y(y, \ln \lambda_a) \psi\left(\frac{\Phi}{\tau}\right) \\ &\quad \frac{4\xi(r - r_a(\tau, \Phi)) - 2\xi\lambda_a^{-2}(\tau^2 W(r - r_a(\tau, \Phi)) + 2(r - M)\xi^2)}{(r - r_a(\tau, \Phi))^2 + \lambda_a^{-2}\xi^2} \end{aligned}$$

where $y = (r - r_a(\tau, \Phi))^2 + \lambda_a^{-2}\xi^2$ and $W \approx r^2$. The first three factors on the right are bounded. The fourth is bounded by $\max\{\lambda_a, \tau\}$ by Cauchy Schwarz and the fact that q is supported in $|\xi| \lesssim \lambda_a$. Hence we obtain $q \in (\tau + \lambda_a) S_{1-\delta, \delta}^{0}$, and the commutator bound (2.32) follows.

Given (2.30) and (2.31), we localize as before ψ to time intervals of unit length and then apply the local Strichartz estimates. By summing over these strips we obtain

$$\|\nabla \psi\|_{L^p H^{-\rho, q}} \lesssim \|u\|_{LEK_{ps}} + \|L_K u\|_{L^2}$$

for all sharp Strichartz pairs (ρ, p, q) .

To return to u we invert B_{ps}^{-1} ,

$$u = B_{ps} \psi + (I - B_{ps} B_{ps}^{-1})u.$$

The second term is much more regular, since by (2.19) we have

$$I - B_{ps} B_{ps}^{-1} \in S_{1-\delta, 0}^{-1+\delta}, \quad \delta > 0.$$

This leads to

$$\|\nabla(I - B_{ps} B_{ps}^{-1})u\|_{L^2 H^{1-\delta}} \lesssim \|u\|_{LEK_{ps}}, \quad \delta > 0;$$

therefore all the Strichartz estimates are satisfied simply by Sobolev embeddings.

For the main term $B_{ps}\psi$ we take advantage of the fact that we only seek to prove the nonsharp Strichartz estimates for u . The nonsharp Strichartz estimates for ψ are obtained from the sharp ones via Sobolev embeddings,

$$\|\nabla\psi\|_{H^{-\rho_2, q_2}} \lesssim \|\nabla\psi\|_{H^{-\rho_1, q_1}}, \quad \frac{3}{q_2} + \rho_2 = \frac{3}{q_1} + \rho_1, \quad \rho_1 < \rho_2.$$

To obtain the nonsharp estimates for u instead, we need a slightly stronger form of the above bound, namely

Lemma 2.7. *Assume that (ρ_1, p_1, q_1) , (ρ_2, p_2, q_2) are Strichartz pairs with $p_1 < p_2$, $q_1 < q_2 < \infty$. Then*

$$(2.33) \quad \|B_{ps}w\|_{L^{p_2}H^{-\rho_2, q_2}} \lesssim \|w\|_{L^{p_1}H^{-\rho_1, q_1}}, \quad \frac{3}{q_2} + \rho_2 = \frac{3}{q_1} + \rho_1.$$

Proof. We need to prove that the operator

$$\tilde{B} = Op^w(\xi^2 + \lambda_0^2 + 1)^{-\frac{\rho_2}{2}} B_{ps} Op^w(\xi^2 + \lambda_0^2 + 1)^{\frac{\rho_1}{2}}$$

maps L^{q_1} into L^{q_2} . The principal symbol of \tilde{B} is

$$\tilde{b}(r, \tau, \xi, \Phi, \lambda_a) = (\xi^2 + \lambda_0^2 + 1)^{\frac{\rho_1 - \rho_2}{2}} b_{ps}(r^*, \xi, \lambda_a),$$

and by the pdo calculus the remainder is easy to estimate,

$$\tilde{B} - \tilde{b}^w \in OPS_{1-\delta, 0}^{\rho_1 - \rho_2 - 1 + \delta}, \quad \delta > 0.$$

The conclusion of the lemma will follow from the Hardy-Littlewood-Sobolev inequality (applied separately for t , r and ω) if we prove a suitable pointwise bound on the kernel K of b_0^w , namely

$$(2.34) \quad |K(t_1, r_1, \omega_1, t_2, r_2, \omega_2)| \lesssim |t_1 - t_2|^{-1 + \frac{1}{p_1} - \frac{1}{p_2}} (|r_1 - r_2| |\omega_1 - \omega_2|^2)^{-1 + \frac{1}{q_1} - \frac{1}{q_2}}.$$

For fixed r we consider a smooth dyadic partition of unity in frequency as follows:

$$1 = (1 - \chi_{\{\tau \approx \lambda_a\}}) \chi_{\{|\xi| > \lambda_a\}} \chi_{\{\mu < 1\}} + \chi_{\{\tau \approx \lambda_a\}} \sum_{\mu \geq 1 \text{ dyadic}} \chi_{\{\lambda_a \approx \mu\}} \sum_{\nu=0}^{\mu} \chi_{\{|\xi| \approx \nu\}}$$

This leads to a similar decomposition for \tilde{b} , namely

$$\tilde{b} = \tilde{b}_{00} + \sum_{\mu \geq 1} \sum_{\nu=0}^{\mu} \tilde{b}_{\mu\nu}.$$

In the region $\{|\xi| \gtrsim \lambda\} \cup \{\mu < 1\} \cup \{\tau \gg \lambda\} \cup \{\tau \ll \lambda\}$ the symbol \tilde{b} is of class $S^{\rho_1 - \rho_2}$, which yields a kernel bound for \tilde{b}_{00} of the form

$$|K_{00}(t_1, r_1, \omega_1, t_2, r_2, \omega_2)| \lesssim (|t_1 - t_2| + |r_1 - r_2| + |\omega_1 - \omega_2|)^{-3 + \rho_2 - \rho_1}.$$

The symbols of $\tilde{b}_{\mu\nu}$ are supported in $\{|\xi| \approx \nu, \lambda \approx \tau \approx \mu\}$, are smooth on the same scale and have size at most $\ln(\nu^{-1}\mu)\mu^{\rho_1 - \rho_2}$. Clearly on their support we also have $\lambda_0 \approx \lambda_a \approx \mu$. Hence after integration by parts we get that their kernels satisfy bounds of the form

$$\begin{aligned} & |K_{\mu, \nu}(t_1, r_1, \omega_1, t_2, r_2, \omega_2)| \\ & \lesssim \ln(\nu^{-1}\mu)\mu^{\rho_1 - \rho_2} \nu (|r_1 - r_2|\nu + 1)^{-N} \mu^2 (|\omega_1 - \omega_2|\mu + 1)^{-N} (|t_1 - t_2|\mu + 1)^{-N} \\ & \lesssim |t_1 - t_2|^{-1 + \frac{1}{p_1} - \frac{1}{p_2}} \ln(\nu^{-1}\mu)\mu^{\frac{3}{q_1} - \frac{3}{q_2}} \nu (|r_1 - r_2|\nu + 1)^{-N} \mu^2 (|\omega_1 - \omega_2|\mu + 1)^{-N} \end{aligned}$$

for all nonnegative integers N . Then (2.34) follows after summation. \square

\square

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